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Constrained Minima of Nonlocal Free Energy Functionals

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We consider variational problems involving nonlocal free energy functionals that arise from Gibbs measures with Kac potentials and are related to the characterization of the optimal (i.e., typical) shape of an interface under given constraints on the magnetization profile.

KEY WORDS: Ising systems; Kac potentials; Γ-convergence; surface tension.

1. INTRODUCTION

This paper continues the analysis of the Gibbs measures with Kac potentials developed in refs. 2 and 1 by focusing on the structure of the interfaces in ($d \geqslant 2$)-dimensional systems. We refer to the companion paper[1] for a general discussion on this issue in the framework of systems with Kac interactions. The main goal in ref. 1 was to compute the probability of observing a given interface in the scaling limit when the range of the Kac potential diverges. Here we do not fix the interface itself, but determine its optimal (i.e., typical) shape under imposed constraints on the magnetization profile. The best-known example (included in our analysis) is the constraint that fixes the value of the total magnetization; the optimal shape is then the Wulff shape. This problem is well known in the literature; see, for instance, ref. 4 for the 2D nearest neighbor ferromagnetic Ising model and ref. 5 for the Ginzburg–Landau functional.

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Our techniques exploit the Γ-convergence result proved in ref. 1 concerning the nonlocal free energy functionals that arise in the continuum limit from Gibbs measures with Kac potentials. The analysis of the constrained minima is then based on a equicoercivity condition, Lemma 2.3, which implies convergence by subsequences.

The outline of the paper is the following: in Section 2 we state the variational problem and in Section 3 its relation with the spin systems, and in Section 4 we prove Lemma 2.3.

2. A VARIATIONAL PROBLEM WITH NONLOCAL FUNCTIONALS

We start with an abstract formulation of the variational problem that we then specialize to the actual case of interest. In the next section we will explain its physical origin, outlining the relation with the spin system.

Let X be a metric space, $\varepsilon > 0$, F_ε and F functions defined on X with values in $[0, +\infty]$, $c(X)$ a subset of all the continuous functions $g: X \to \mathbb{R}$ such that $g^{-1}(0) \neq \emptyset$.

Definition 2.1. We say that $\{F_\varepsilon\}$ Γ-converges to F under the constraint g if

$$\lim_{\zeta \to 0^+} \liminf_{\varepsilon \to 0^+} \inf_{|g(u)| < \zeta} F_\varepsilon(u) = \lim_{\zeta \to 0^+} \limsup_{\varepsilon \to 0^+} \inf_{|g(u)| < \zeta} F_\varepsilon(u) = \inf_{g(u) = 0} F(u) \tag{2.1}$$

We also say that $\{F_\varepsilon\}$ Γ-converges to F under the set of constraints $c(X)$ if it Γ-converges for any $g \in c(X)$.

If the set $c(X)$ consists of all the functions g_v, $v \in X$, where

$$g_v(u) = \delta(u, v) \tag{2.2}$$

and δ is the distance on X, then the first two terms in (2.1) are respectively the usual Γ-lower and upper limits of the sequence $\{F_\varepsilon\}$ at the point v, and F is the Γ-limit of $\{F_\varepsilon\}$ on X (see Chapter 3 in ref. 3).

In our applications $X = L^1(\mathscr{T}; [-1, 1])$, where $\mathscr{T}$ is the unit torus in $\mathbb{R}^d$. The elements u of $L^1(\mathscr{T}; [-1, 1])$ are interpreted as "magnetization profiles" and $u(r)$ as the magnetization density at the "macroscopic space point" $r \in \mathscr{T}$. The number $\varepsilon > 0$, which for simplicity we choose so that $\varepsilon^{-1} \in \mathbb{N}$, is a scaling parameter which represents the ratio between macroscopic and microscopic units; the limit $\varepsilon \to 0^+$ thus corresponds to the macroscopic limit. The quantity $F_\varepsilon(u)$ [see (2.4) below] is the (excess) free energy of the magnetization profile u corresponding to the scaling parameter ε. The functions g are the physical observables; thus, fixing $|g(u)| < \zeta$ amounts to preparing the system in a state where the observable g has value 0 with "tolerance" ζ. The state that will then be selected by the system in such a condition is the one that minimizes the free energy under the given constraint.

Using (2.1) with $g = g_v$, $v \in L^1(\mathscr{T}; [-1, 1])$ [see (2.2)], we obtain the limit free energy $F(v)$ of the profile v. Observe that in general $F(v)$ will be different from the limit of the free energies $F_\varepsilon(v)$ since Γ-convergence and pointwise convergence of $\{F_\varepsilon\}$ do not necessarily coincide. Physically this is due to the fact that the same macroscopic state v can be realized in many, slightly different (with tolerance ζ) microscopic states, differences that in macroscopic units vanish when $\varepsilon \to 0^+$. However, for each $\varepsilon > 0$ the system "is free to look around" and choose among all these states the one with the lowest free energy. This effect may persist in the macroscopic limit, no matter how precise is the preparation of the system (i.e., ζ small).

In interface problems a typical constraint is

$$g(u) := \int_{\mathscr{T}} dr\, u(r) - a, \qquad a \in [-1, 1] \tag{2.3}$$

and the associated variational problem consists in finding the minimal free energy at the given magnetization a (with tolerance ζ), which is a Wulff-type problem. In $d = 2$ Ising nearest neighbor interactions this problem was first solved in ref. 4; the result has been extended in several directions. For Ginzburg–Landau functionals the Wulff problem has been solved in ref. 6; see also ref. 5.

The free energy functions F_ε in which we are interested read

$$F_\varepsilon(u) := \varepsilon^{-1} \int_{\mathscr{T}} dr\, \omega(u(r)) + \frac{\varepsilon}{4} \iint_{\mathscr{T} \times \mathscr{T}} dr\, dr'\, J_\varepsilon(|r - r'|) \left[\frac{u(r) - u(r')}{\varepsilon} \right]^2 \tag{2.4}$$

where $u \in L^1(\mathscr{T}; [-1, 1])$, and ω is a double-well potential defined by

$$\omega(s) := f(s) - f(m_\beta), \qquad s \in [-1, 1] \tag{2.5}$$

with m_β the positive solution of the mean-field equation

$$m_\beta = \tanh(\beta m_\beta) \tag{2.6}$$

and f the free energy density. Precisely

$$f(s) := -\tfrac{1}{2} s^2 - \beta^{-1} i(s) \tag{2.7}$$

where $\beta > 1$ is the inverse temperature and i is the entropy density, i.e.,

$$i(s) := -\frac{1+s}{2}\log\left(\frac{1+s}{2}\right) - \frac{1-s}{2}\log\left(\frac{1-s}{2}\right) \tag{2.8}$$

Finally

$$J_\varepsilon(|r|) := \varepsilon^{-d} J(\varepsilon^{-1}|r|) \tag{2.9}$$

with $J(|r|)$ (the interaction strength) a nonnegative $\mathscr{C}^\infty$ function of $r \in \mathbb{R}^d$, supported in the unit ball and such that

$$\int_{\mathbb{R}^d} dr\, J(|r|) = 1 \tag{2.10}$$

The expression (2.4) recalls the Ginzburg–Landau free energy functional, of which it is a nonlocal version; see the end of Section 2 and the beginning of Section 4 in ref. 1. The Γ-convergence of the Ginzburg–Landau functionals has been proved in ref. 5, while in ref. 1 it is shown that the functions F_ε in (2.4) Γ-converge to $F := s_\beta P$ on $BV(\mathscr{T}; \{\pm m_\beta\})$, where $BV(\mathscr{T}; \{\pm m_\beta\})$ is the space of functions of bounded variations on $\mathscr{T}$ with values $\pm m_\beta$. Here $s_\beta > 0$ is the surface tension at the inverse temperature β, and $P(v)$ is the generalized area of the boundary of the set $\{v(r) = m_\beta\}$. This shows the validity of (2.1) for all g_v with $v \in BV(\mathscr{T}; \{\pm m_\beta\})$; the extension to all g is proved in the next theorem. We first extend F by setting

$$F(v) := \begin{cases} s_\beta P(v) & \text{for } v \in BV(\mathscr{T}; \{\pm m_\beta\}) \\ +\infty & \text{otherwise on } L^1(\mathscr{T}; [-1, 1]) \end{cases}$$

and then state the main result in this paper:

Theorem 2.2. Let $X = L^1(\mathscr{T}; [-1, 1])$. The sequence $\{F_\varepsilon\}$ defined in (2.4) Γ-converges to F under any constraint, namely (2.1) holds for all continuous functions $g: X \to \mathbb{R}$ such that $g^{-1}(0) \neq \emptyset$.

In particular, Theorem 2.2 proves that $\{F_\varepsilon\}$ Γ-converges to F on the whole $L^1(\mathscr{T}; [-1, 1])$. Thus if $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ and $u \in L^1(\mathscr{T}; [-1, 1]) \setminus BV(\mathscr{T}; \{\pm m_\beta\})$, then $F_\varepsilon(u_\varepsilon) \to +\infty$.

The proof of Theorem 2.2 is based on the following compactness condition, which is the main technical estimate in the paper and will be proved in Section 4:

Lemma 2.3. Let $\{u_\varepsilon\}$ be a sequence in $L^1(\mathscr{T}; [-1, 1])$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Then $\{u_\varepsilon\}$ admits a subsequence converging in $L^1(\mathscr{T})$ to a function in $BV(\mathscr{T}; \{\pm m_\beta\})$.

Proof of Theorem 2.2. Set $X := L^1(\mathscr{T}; [-1, 1])$, $X' := BV(\mathscr{T}; \{\pm m_\beta\})$. Fix $g \in c(X)$; given $\varepsilon > 0$ and $\zeta > 0$, let

$$I_\varepsilon(\zeta) := \inf_{|g(u)| < \zeta} F_\varepsilon(u), \qquad I(\zeta) := \inf_{|g(u)| < \zeta} F(u), \qquad I(0) := \inf_{g(u) = 0} F(u)$$

We first consider the case $I(0) < +\infty$. We shall prove that

$$I(2\zeta) \leqslant \liminf_{\varepsilon \to 0^+} I_\varepsilon(\zeta) \leqslant \limsup_{\varepsilon \to 0^+} I_\varepsilon(\zeta) \leqslant I(\zeta) \tag{2.11}$$

Given any $\delta > 0$, there is $u^\delta \in X$ so that

$$|g(u^\delta)| < \zeta, \qquad F(u^\delta) < I(\zeta) + \delta \tag{2.12}$$

Since $I(\zeta) \leqslant I(0) < +\infty$, we get $P(u^\delta) < +\infty$, hence $u^\delta \in X'$.

As $\{F_\varepsilon\}$ Γ-converges to F on X', there is a sequence $\{u_\varepsilon\}$ such that $u_\varepsilon \to u^\delta$ in X and $F_\varepsilon(u_\varepsilon) \to F(u^\delta)$ as $\varepsilon \to 0^+$. By the continuity of g, if $\varepsilon > 0$ is small enough, we have $|g(u_\varepsilon)| < \zeta$, hence $I_\varepsilon(\zeta) \leqslant F_\varepsilon(u_\varepsilon)$. By (2.12) we then have

$$\limsup_{\varepsilon \to 0^+} I_\varepsilon(\zeta) \leqslant \lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) = F(u^\delta) \leqslant I(\zeta) + \delta$$

and letting $\delta \to 0$, we deduce that

$$\limsup_{\varepsilon \to 0^+} I_\varepsilon(\zeta) \leqslant I(\zeta) \tag{2.13}$$

Let now $\{u_\varepsilon\}$ be a sequence in X such that $|g(u_\varepsilon)| < \zeta$ and

$$\liminf_{\varepsilon \to 0^+} I_\varepsilon(\zeta) = \lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \tag{2.14}$$

As $I(\zeta) \leqslant I(0) < +\infty$, by (2.13) it follows that $\lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) < +\infty$. Using Lemma 2.3, there exist a function $u \in X'$ and a subsequence $\{u_{\varepsilon'}\}$ of $\{u_\varepsilon\}$ converging to u in $L^1(\mathscr{T})$. But $\{F_\varepsilon\}$ Γ-converges to F on $BV(\mathscr{T}; \{\pm m_\beta\})$ and $|g(u)| \leqslant \zeta$ because g is continuous, hence

$$\liminf_{\varepsilon \to 0^+} I_\varepsilon(\zeta) = \lim_{\varepsilon' \to 0^+} F_{\varepsilon'}(u_{\varepsilon'}) \geqslant F(u) \geqslant I(2\zeta) \tag{2.15}$$

The inequalities in (2.11) follow from (2.13) and (2.15).

Moreover,

$$\limsup_{\zeta \to 0^+} I(\zeta) \leqslant I(0)$$

and using the facts that F is coercive and g is continuous, one can check that

$$\liminf_{\zeta \to 0^+} I(\zeta) \geqslant I(0)$$

Hence, passing to the limit as $\zeta \to 0^+$ in (2.11), we deduce (2.1).

It remains to prove (2.1) when $I(0) = +\infty$. We need to prove that

$$\lim_{\zeta \to 0^+} \liminf_{\varepsilon \to 0^+} \inf_{|g(u)| < \zeta} F_\varepsilon(u) = +\infty \tag{2.16}$$

Suppose by contradiction that there is a sequence $\{u_\varepsilon\}$ in X such that

$$\lim_{\varepsilon \to 0^+} g(u_\varepsilon) = 0, \qquad \sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty \tag{2.17}$$

By Lemma 2.3, $\{u_\varepsilon\}$ admits a subsequence converging (in X) to $u \in X'$, so that $F(u) < +\infty$. As g is continuous, we have $g(u) = 0$, hence $I(0) < +\infty$. Theorem 2.2 is proved. ∎

The proof of Theorem 2.2 applies to more general situations, as in the case in which X' is a proper subset of the metric space X, $F^{-1}(+\infty) = X \backslash X'$, $\{F_\varepsilon\}$ Γ-converges to F on X', and the following conditions hold: if $t \in [0, +\infty)$ and either $u_\varepsilon \in \{F \leqslant t\}$ or $u_\varepsilon \in \{F_\varepsilon \leqslant t\}$ for any $\varepsilon > 0$, then $\{u_\varepsilon\}$ admits a subsequence converging in X to some element of X'.

3. APPLICATIONS TO ISING SYSTEMS WITH KAC POTENTIALS

We start with some notation (Definition 3.1 below) that will be useful also in Section 4, then we will recall from ref. 1 the basic definitions of Ising systems with (ferromagnetic) Kac potentials and state without proofs the analogue of Theorem 2.2 at the spin level.

Definition 3.1. Partitions of $\mathbb{R}^d$. For any $k \in \mathbb{Z}$, $\mathcal{Q}^{(k)}$ denotes the partition of $\mathbb{R}^d$ into the d-dimensional cubes

$$\{r = (r_1, \dots, r_d) \in \mathbb{R}^d : 2^{-k} x_i \leqslant r_i < 2^{-k}(x_i + 1);\, x_i \in \mathbb{Z},\ i = 1, \dots, d\} \tag{3.1}$$

The atoms of $\mathcal{Q}^{(k)}$ are denoted by $C^{(k)}$. Here $C^{(k)}(r)$ is the unique atom of $\mathcal{Q}^{(k)}$ that contains the point r. We let $\pi^{(k)}$ be the map from $L^\infty(\mathbb{R}^d)$ into itself defined by

$$\pi^{(k)} f(r) := \frac{1}{|C^{(k)}|} \int_{C^{(k)}(r)} dr'\, f(r') \tag{3.2}$$

A function $f \in L^\infty(\mathbb{R}^d)$ is $\mathscr{Q}^{(k)}$-measurable if $f = \pi^{(k)} f$ and a set $A \subset \mathbb{R}^d$ is $\mathscr{Q}^{(k)}$-measurable if its characteristic function $\mathbf{1}_A$ is $\mathscr{Q}^{(k)}$-measurable.

Definition 3.2. Spin configurations. We denote by γ a parameter that takes values in $\{2^{-k}, k \in \mathbb{N}\}$. Then σ_γ is an Ising spin configuration with mesh $\gamma = 2^{-k_\gamma}$, $k_\gamma \in \mathbb{N}$, if $\sigma_\gamma \in L^\infty(\mathbb{R}^d; \{\pm 1\})$ and if σ_γ is $\mathscr{Q}^{(k_\gamma)}$-measurable.

Let $\alpha \in (0, (d+1)^{-1})$, $\varepsilon^{-1} := [\gamma^{-\alpha}]$ ($[a]$ denoting the integer part of $a \in \mathbb{R}$), $\mathscr{T}_\varepsilon$ the torus in $\mathbb{R}^d$ of period ε^{-1}. A spin configuration on $\mathscr{T}_\varepsilon$ is an Ising spin configuration σ_γ with mesh γ periodic with period ε^{-1}. Hereafter σ_γ will always denote a spin configuration on $\mathscr{T}_\varepsilon$.

Definition 3.3. Energy. Let Λ be a bounded measurable region of $\mathbb{R}^d$ and $m \in L^1(\Lambda; [-1, 1])$. The energy of m in Λ is defined as

$$H(m; \Lambda) := -\tfrac{1}{2} \int_\Lambda dr \int_\Lambda dr'\, J(|r - r'|)\, m(r)\, m(r') \tag{3.3}$$

If Λ is a torus, then $|r - r'|$ in (3.3) is the distance between r and r' in the torus.

Definition 3.4. Gibbs measures. The Gibbs measure on the torus $\mathscr{T}_\varepsilon$, with Kac potential $J(|r|)$, scaling parameter γ, and inverse temperature $\beta > 1$, is the probability $\mu_{\beta, \gamma, \varepsilon}$ on the space of spin configurations on $\mathscr{T}_\varepsilon$ with mesh γ defined as

$$\mu_{\beta, \gamma, \varepsilon}(\sigma_\gamma) := \frac{1}{Z_{\beta, \gamma, \varepsilon}} e^{-\beta \gamma^{-d} H(\sigma_\gamma; \mathscr{T}_\varepsilon)} \tag{3.4}$$

where

$$Z_{\beta, \gamma, \varepsilon} := \sum_{\sigma_\gamma} e^{-\beta \gamma^{-d} H(\sigma_\gamma; \mathscr{T}_\varepsilon)} \tag{3.5}$$

is the partition function.

See ref. 1 for a discussion on the above definitions and their relation to more usual formulations of the model.

Given $m \in L^1(\mathscr{T}_\varepsilon; [-1, 1])$ and $\varepsilon > 0$, we denote by $m_\varepsilon \in L^1(\mathscr{T}; [-1, 1])$, $\mathscr{T}$ the unit torus in $\mathbb{R}^d$, the function

$$m_\varepsilon(r) := m(\varepsilon^{-1} r) \tag{3.6}$$

Let g be a constraint on $X := L^1(\mathscr{T}; [-1, 1])$, $\zeta > 0$, and $k \in \mathbb{N}$. We want to study the behavior of

$$\mu_{\beta, \gamma, \varepsilon}(\{ |g((\pi^{(k)} \sigma_\gamma)_\varepsilon)| < \zeta \}) \tag{3.7}$$

as $\gamma \to 0^+$, $\varepsilon \to 0^+$, and then $\zeta \to 0^+$. In fact (3.7) is the probability that the observed value of g is 0 with tolerance ζ when g is computed on the block spin configuration $\pi^{(k)}\sigma_\gamma$.

We omit, for reasons of space, the proof (consequence of Theorem 2.2 and of the analysis in Section 3 of ref. 1) that if $\varepsilon^{-1} \approx \gamma^{-\alpha}$ with $\alpha > 0$ small enough (see Definition 2.1e in ref. 1), then

$$\begin{aligned}
&\lim_{\zeta \to 0^+} \liminf_{\gamma \to 0^+} \gamma^d \varepsilon^{d-1} \ln \mu_{\beta,\gamma,\varepsilon}(\{ |g(M_\varepsilon \pi^{(k)}\sigma_\gamma)| < \zeta \}) \\
&\quad = \lim_{\zeta \to 0^+} \limsup_{\gamma \to 0^+} \gamma^d \varepsilon^{d-1} \ln \mu_{\beta,\gamma,\varepsilon}(\{ |g(M_\varepsilon \pi^{(k)}\sigma_\gamma)| < \zeta \}) \\
&\quad = -\beta s_\beta \inf_{g(u)=0} P(u)
\end{aligned} \tag{3.8}$$

4. PROOF OF LEMMA 2.3

Given $\varepsilon > 0$ and $m \in L^\infty(\mathcal{T}_\varepsilon; [-1, 1])$, we call

$$m^{(1)} := \begin{cases} m_\beta & \text{on } \{m \geqslant m_\beta\} \\ -m_\beta & \text{on } \{m \leqslant -m_\beta\} \\ m & \text{elsewhere on } \mathcal{T}_\varepsilon \end{cases} \tag{4.1}$$

Recalling Definition 3.1, we define the function $\phi = \phi_{m,k,\zeta}$, $m \in L^\infty(\mathcal{T}_\varepsilon; [-m_\beta, m_\beta])$, $k \in \mathbb{N}_+$, $\zeta > 0$, as

$$\phi(r) := \begin{cases} 1 & \text{if } \pi^{(k)}m(r) \geqslant m_\beta - \zeta \\ -1 & \text{if } \pi^{(k)}m(r) \leqslant -m_\beta + \zeta \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

Finally, given $m \in L^\infty(\mathcal{T}_\varepsilon; [-1, 1])$, $k \in \mathbb{N}_+$, and $\zeta > 0$, we define the function $m^+ \in L^\infty(\mathcal{T}_\varepsilon; \{\pm m_\beta\})$ by setting $\phi := \phi_{m^{(1)},k,\zeta}$ and

$$m^+ := \begin{cases} m_\beta & \text{on } \{\phi \geqslant 0\} \\ -m_\beta & \text{on } \{\phi = -1\} \end{cases} \tag{4.3}$$

We also denote by $P(u; \mathcal{T}_\varepsilon)$ the perimeter functional for $u \in BV(\mathcal{T}_\varepsilon; \{\pm m_\beta\})$ and define $\mathcal{F}(m; r_\varepsilon) := \varepsilon^{1-d} F_\varepsilon((m)_\varepsilon)$, see (3.6).

Proposition 4.1. There are $k \in \mathbb{N}_+$ and $\zeta > 0$ and for any $t > 0$ there is $c > 0$ so that the following holds. Given $\varepsilon > 0$, if $m \in L^\infty(\mathcal{T}_\varepsilon; [-1, 1])$ is such that $\mathcal{F}(m, \mathcal{T}_\varepsilon) \leqslant t\varepsilon^{1-d}$, then m^+ as defined in (4.3) satisfies

$$P(m^+; \mathcal{T}_\varepsilon) \leqslant c\mathcal{F}(m; \mathcal{T}_\varepsilon) \tag{4.4}$$

$$\int_{\mathcal{T}_\varepsilon} dr\, |m^+(r) - m(r)|^2 \leqslant c\mathcal{F}(m; \mathcal{T}_\varepsilon) \tag{4.5}$$

Proof. Let $k \in \mathbb{N}_+$, $\zeta > 0$, $\varepsilon > 0$, $m \in L^\infty(\mathcal{T}_\varepsilon; [-1, 1])$, and $\phi = \phi_{m^{(1)}, k, \zeta}$ as in (4.2). Let ∂^+ be the boundary (in $\mathcal{T}_\varepsilon$) of the set $\{m^+ = m_\beta\}$. By the definition of ϕ, ∂^+ is made up of faces of elements of $\mathcal{Q}^{(k)}$. We can then write ∂^+ as the union of ∂_i^+, $i \in I$, I a finite index set, where each ∂_i^+ is one of the faces of some $C^i \in \mathcal{Q}^{(k)}$, C^i chosen in the following way: either $\phi = 0$ on C^i or $\partial_i^+ = \partial C^i \cap \partial C$ for some $C \in \mathcal{Q}^{(k)}$, where $\phi = 1$ on C^i and $\phi = -1$ on C, or vice versa.

For any C^i we denote by D^i the cube of side 4 with same center as C^i. In the proof of Theorem 2.3 of ref. 1 it is shown that if k is large enough and ζ small enough, then there is $c > 0$ (independent of ε) so that for all $i \in I$

$$|\partial C^i| \leqslant c\mathcal{F}(m^{(1)}; D^i) \tag{4.6}$$

Hence

$$P(m^+; \mathcal{T}_\varepsilon) = |\partial^+| = \sum_{i \in I} |\partial C^i| \leqslant c \sum_{i \in I} \mathcal{F}(m^{(2)}; D^i) \tag{4.7}$$

Observing that if A and B are disjoint sets in $\mathcal{T}_\varepsilon$, then

$$\mathcal{F}(\cdot\,; A \cup B) \geqslant \mathcal{F}(\cdot\,; A) + \mathcal{F}(\cdot\,; B)$$

we would get (4.4) from (4.7) if the D^i were disjoint. Since $D^i \cap D^j = \varnothing$ if the distance between C^i and C^j is larger than 4, the number of disjoint D^i is at least $(5 \cdot 2^k)^{-d}$ times the number of cubes C^i. Therefore

$$P(m^+; \mathcal{T}_\varepsilon) \leqslant c\, 2^{(k+3)d} \mathcal{F}(m^{(2)}; \mathcal{T}_\varepsilon) \leqslant c\, 2^{(k+3)d} \mathcal{F}(m; \mathcal{T}_\varepsilon) \tag{4.8}$$

which proves (4.4).

Let us prove (4.5). We first assume that $m \in L^\infty(\mathcal{T}_\varepsilon; [-m_\beta, m_\beta])$, so that $m = m^{(1)}$.

$$\int_{\mathcal{T}_\varepsilon} dr\, |m^+(r) - m(r)|^2$$

$$= \sum_{C \in \mathcal{Q}^{(k)} \cap \mathcal{T}_\varepsilon} \int_C dr\, |m^+(r) - m(r)|^2 =: \Sigma_0 + \Sigma_1 \tag{4.9}$$

where Σ_0 is the sum over all $C \subseteq \{\phi = 0\}$ and Σ_1 is the sum over all the others. If $C \subseteq \{\phi = 0\}$, by (4.6) we have for a suitable constant $c > 0$

$$\int_C dr\, |m^+(r) - m(r)|^2 \leqslant 4\, |C| \leqslant 4 \frac{|C|}{|\partial C|} c\mathscr{F}(m^{(1)}; D)$$

$$= 4 \frac{|C|}{|\partial C|} c\mathscr{F}(m; D) \tag{4.10}$$

so that for some new constant $c > 0$

$$\Sigma_0 \leqslant c\mathscr{F}(m; \mathscr{T}_\varepsilon) \tag{4.11}$$

Assume now that $C \subseteq \{\phi \neq 0\}$; we shall suppose that $C \subseteq \{\phi = 1\}$, the other case being similar. Since the function $\omega(s)$, $s \in \mathbb{R}$, is a symmetric double-well function with two quadratic minima at $s = \pm m_\beta$, given $\zeta \in (0, 3m_\beta/2)$, there is $c > 0$ so that

$$(s - m_\beta)^2 \leqslant c\omega(s) \qquad \text{for all} \quad s \in [-m_\beta + \zeta, m_\beta] \tag{4.12}$$

We write $C = C^+ \cup C_2$,

$$C^+ := \{r \in C: m(r) \geqslant -m_\beta + \zeta\}$$

$$C_2 := C \backslash C^+ = \{r \in C: m(r) < -m_\beta + \zeta\}$$

Recalling that $m^+ = m_\beta$ on C and using (4.12), we obtain

$$\int_{C^+} dr\, |m(r) - m^+(r)|^2 \leqslant c \int_{C^+} dr\, [f(m(r)) - f(m_\beta)] \leqslant c\mathscr{F}(m; D) \tag{4.13}$$

(this is the only place where the L^2-norm of $m - m^+$ is essential). It thus remains to estimate the integral over C_2, where, however, an inequality like the first one in (4.13) cannot hold in general [as when $m(r) = -m_\beta$ for $r \in C_2$; then $|m(r) - m^+(r)| = 2m_\beta$ in C_2, while $\omega(m(r)) = 0$]. We will then use for the bound in C_2 the second term in the functional $\mathscr{F}$. To this end we define

$$C_1 := \{r \in C: m(r) \geqslant m_\beta - 2\zeta\}, \qquad C_0 := C \backslash (C_1 \cup C_2) \tag{4.14}$$

We have $C_1 \subset C^+$ so that $C_1 \cap C_2 = \varnothing$. Since $\phi = 1$ on C, the average of m over C is bounded from below by $m_\beta - \zeta$. Hence

$$
\begin{aligned}
(m_\beta-\zeta)\,|C| &= (m_\beta-\zeta)(|C_0|+|C_1|+|C_2|) \\
&\leqslant \int_{C_0} dr\, m(r) + \int_{C_1} dr\, m(r) + \int_{C_2} dr\, m(r) \\
&\leqslant (m_\beta-2\zeta)\,|C_0| + m_\beta\,|C_1| + (-m_\beta+\zeta)\,|C_2|
\end{aligned}
$$

so that

$$
2\,|C_2|\,(m_\beta-\zeta) \leqslant -|C_0|\,\zeta + |C_1|\,\zeta \leqslant |C_1|\,\zeta
$$

Hence, for $\zeta>0$ sufficiently small, we have $|C_2|\leqslant|C_1|$. By the isomorphism of Lebesgue measures,[7] there is a measurable subset A_1 of C_1 and a one-to-one map ψ from C_2 onto A_1 which preserves the Lebesgue measure. For k large enough there are a cube $C'\in\mathscr{Q}^{(k)}$, $C'\subset D\backslash C$, and $a>0$ such that $J(|r-r'|)\geqslant a$ for all $r\in C$ and $r'\in C'$. We can then bound

$$
\mathscr{F}(m; D) \geqslant \frac{a}{4}\int_{C'} dr' \int_{A_1\cup C_2} dr\,[m(r')-m(r)]^2 \tag{4.15}
$$

We write the integral over $r\in A_1\cup C_2$ as

$$
\begin{aligned}
&\int_{C_2} dr\,\{[m(r')-m(r)]^2+[m(r')-m(\psi(r))]^2\} \\
&\qquad \geqslant \frac{1}{2}\int_{C_2} dr\,[m(\psi(r))-m(r)]^2 \geqslant \frac{|C_2|}{2}(2m_\beta-3\zeta)^2
\end{aligned}
$$

where we have used the elementary inequality

$$
\frac{1}{2}((z-x)^2+(x-y)^2) \geqslant \left(\frac{z-y}{2}\right)^2
$$

and the facts that $r\in C_2$ implies $m(r)<-m_\beta+\zeta$ and $\psi(r)\in A_1\subset C_1$ implies $m(\psi(r))\geqslant m_\beta-2\zeta$. Using (4.15), we then have

$$
\begin{aligned}
\mathscr{F}(m; D) &\geqslant \frac{a}{4}\,|C'|\,\frac{(2m_\beta-3\zeta)^2}{2}\,|C_2| \\
&\geqslant \frac{a}{4}\,2^{-kd}\,\frac{(2m_\beta-3\zeta)^2}{2}\int_{C_2} dr\,\frac{|m(r)-m_\beta|^2}{4}
\end{aligned}
$$

Hence, for a suitable constant $c>0$,

$$
\int_{C_2} dr\,|m(r)-m_\beta|^2 \leqslant c\mathscr{F}(m; D)
$$

and this, toghether with (4.13), yields

$$\int_C dr\, |m^+(r) - m(r)|^2 \leqslant c\mathscr{F}(m; D)$$

so that for a new constant $c > 0$

$$\Sigma_1 \leqslant c\mathscr{F}(m; \mathscr{T}_\varepsilon) \tag{4.16}$$

Inequalities (4.16) and (4.11) prove (4.5) for $m = m^{(1)}$. When $m \neq m^{(1)}$ we write, recalling (4.1),

$$\frac{1}{2}\int_{\mathscr{T}_\varepsilon} dr\, |m^+(r) - m(r)|^2 \leqslant \int_{\mathscr{T}_\varepsilon} dr\, |m^+(r) - m^{(1)}(r)|^2 + \int_{\mathscr{T}_\varepsilon} dr\, |m(r) - m^{(1)}(r)|^2$$

By (4.5) the first integral is bounded by $c\mathscr{F}(m^{(2)}; \mathscr{T}_\varepsilon)$, hence by $c\mathscr{F}(m; \mathscr{T}_\varepsilon)$. By (4.12) the second integral is also bounded by $c\mathscr{F}(m; \mathscr{T}_\varepsilon)$ and Proposition 4.1 is proved. ▌

Proof of Lemma 2.3. Let $t > 0$ be such that $F_\varepsilon(u_\varepsilon) \leqslant t$ for any $\varepsilon > 0$. We reformulate Proposition 4.1 in terms of functions defined on $\mathscr{T}$. Set $u := (m)_\varepsilon$ [i.e., $u(r) = m(\varepsilon^{-1} r)$, $r \in \mathscr{T}$] and $u^+ := (m)_\varepsilon^+$. Recalling that P is the perimeter functional on $BV(\mathscr{T}; \{\pm m_\beta\})$, by Proposition 4.1 we have [writing $|u|_1$ for the $L^1(\mathscr{T})$-norm of u]

$$P(u_\varepsilon^+) \leqslant ct, \qquad |u_\varepsilon^+ - u_\varepsilon|_1 \leqslant \int_{\mathscr{T}} dr\, |u_\varepsilon^+(r) - u_\varepsilon(r)|^2 \leqslant ct\varepsilon \tag{4.17}$$

Since $|u_\varepsilon^+|_1 \leqslant c$, the first inequality in (4.17) implies that there are $u \in BV(\mathscr{T}; \{\pm m_\beta\})$ and a subsequence $\{u_{\varepsilon'}^+\}$ of $\{u_\varepsilon^+\}$ converging to u in $L^1(\mathscr{T})$. Finally, thanks to the last inequality in (4.17), we have

$$\lim_{\varepsilon' \to 0^+} |u_{\varepsilon'} - u|_1 \leqslant \lim_{\varepsilon' \to 0^+} (|u_{\varepsilon'} - u_{\varepsilon'}^+|_1 + |u_{\varepsilon'}^+ - u|_1) = 0 \tag{4.18}$$

and Lemma 2.3 is proved. ▌

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